Sets avoiding norm 1 in $\mathbb{R}^n$

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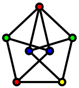
The chromatic number of the plane

- The Hadwiger-Nelson problem 1950: What is the least number of colors needed to color $\mathbb{R}^2$ such that two points at Euclidean distance 1 receive different colors?
- In 1950 Nelson introduced this number $\chi(\mathbb{R}^2)$ and together with Isbell proved that:

$$4 \leq \chi(\mathbb{R}^2) \leq 7$$



- On April 8, 2018, Aubrey de Grey posted on arXiv a paper proving $\chi(\mathbb{R}^2) \geq 5$. He constructs a unit distance graph in the plane with chromatic number 5 and with 1585 vertices (independently verified with SAT solvers).

The unit distance graph on $\mathbb{R}^n$

- The unit distance graph has vertices $\mathbb{R}^n$ and edges $\{x, y\}$ where $\|x - y\| = 1$
- Its chromatic number is denoted $\chi(\mathbb{R}^n)$.
- Its independent sets are sets avoiding distance 1.

  $$A \subset \mathbb{R}^n \text{ avoids distance 1 if } \|x - y\| \neq 1 \text{ for all } x, y \text{ in } A.$$

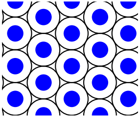
  Example: the color classes of an admissible coloring.
- If A is measurable, it has a (upper) density $\delta(A)$. Let

  $$m_1(\mathbb{R}^n) := \sup\{\delta(A),\ A \text{ measurable, avoids distance } 1\}$$
- For the measurable chromatic number $\chi_m(\mathbb{R}^n)$, the color classes are assumed to be measurable and we have:

  $$\chi_m(\mathbb{R}^n) \geq \frac{1}{m_1(\mathbb{R}^n)}.$$
- Obviously $\chi_m(\mathbb{R}^n) \geq \chi(\mathbb{R}^n)$. Falconer 1981: $\chi_m(\mathbb{R}^n) \geq n + 3$ for all $n \geq 2$.

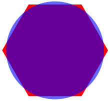
Sets avoiding distance 1

- A set avoiding distance 1 in the plane: open disks of diameter 1, whose centers are hexagonal lattice points with pairwise minimal distance 2.



$$\delta(A) = \frac{\Delta_2}{4} = \frac{\pi}{8\sqrt{3}} \approx 0.226$$

- Best known lower bound for $m_1(\mathbb{R}^2)$: Croft 1967
  Hexagonal arrangement of tortoises (disks cut out by hexagons) of density $\delta \approx 0.229$



The combinatorial upper bounds for $m_1(\mathbb{R}^n)$

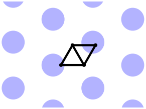
- Let $G = (V, E)$ a finite subgraph embedded in $\mathbb{R}^n$ (ie the edges are the pairs of vertices at distance 1 apart).
  Let $\alpha(G)$ be its independence number.

$\alpha(G) = 2$

- We have:

$$m_1(\mathbb{R}^n) \leq \frac{\alpha(G)}{|V|}$$

Proof: let A be a subset of $\mathbb{R}^n$ avoiding 1. A translated copy of G has on average $|V|\delta(A)$ vertices in A, but also at most $\alpha(G)$ vertices in A.



- Larman Rogers 1972: good graphs for small dimensions.
  Improved by Szekely Wormald 1989.
  Frankl Wilson 1981, Raigorodskii 2000:

$$m_1(\mathbb{R}^n) \lessapprox (1.239)^{-n}$$

The eigenvalue upper bound for $m_1(\mathbb{R}^n)$

- Oliveira, Vallentin 2010: if ω is the normalized surface measure of S^{n-1},

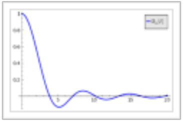
$$m_1(\mathbb{R}^n) \leq \frac{-\min \widehat{\omega}(u)}{1 - \min \widehat{\omega}(u)}$$

- This is a continuous analog of Hoffman bound for finite d-regular graphs:

$$\frac{\alpha(G)}{|V|} \leq \frac{-\lambda_{\min}(A_G)}{d - \lambda_{\min}(A_G)}$$

- The Fourier transform of the surface measure on S^{n-1} expresses in terms of the Bessel function $J_{n/2-1}$:

$$\widehat{\omega}(u) = \Omega_n(\|u\|) = \Gamma(n/2)(2/\|u\|)^{n/2-1} J_{n/2-1}(\|u\|)$$



The eigenvalue upper bound for $m_1(\mathbb{R}^n)$

- Asymptotically the eigenvalue bound is not as good as the combinatorial bound:

$$\frac{-\min \widehat{\omega}(u)}{1-\min \widehat{\omega}(u)} \approx (\sqrt{e/2})^{-n} \approx (1.165)^{-n}$$

- It can be strengthened through extra constraints, leading to the best known bounds for $2 \leq n \leq 24$.

  Oliveira Vallentin 2018: A general framework using the cone of boolean quadratic constraints (Fernando's talk last monday).

- Combined with combinatorial constraints it also leads to: [B., Passuello, Thiery 2013]

$$m_1(\mathbb{R}^n) \lessapprox (1.268)^{-n}$$

Non euclidean norms

- What about other norms?
  In particular what about norms defined by a convex symmetric polytope P?
- Examples: $\| \|_\infty$corresponds to the hypercube; $\| \|_1$corresponds to the crosspolytope.
- In general, $\| \|_P$ is defined by

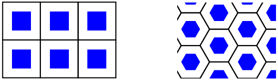
$$\|x\|_P = \min\{\lambda \mid x \in \lambda P\}$$

  and we have similar notions of $m_P(\mathbb{R}^n)$, $\chi_P(\mathbb{R}^n)$.
- The 1-avoiding problem appears to be related to (and more difficult than) the packing problem, in particular if there is a tight packing. Maybe it is easier to analyze if the packing problem is easy or even trivial.

Polytopes that tile space

- If the polytope P tiles space by translations then the density $1/2^n$ is attained by

$$A = \cup_{x \in L}(x + P/2)$$



Conjecture

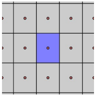
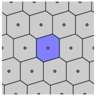
(B., Sinai Robins) If P is a convex polytope that tiles $\mathbb{R}^n$ by translations then

$$m_P(\mathbb{R}^n) = 1/2^n$$

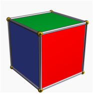
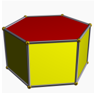
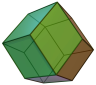
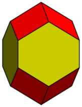
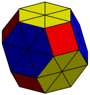
- The 2^n translates of A provide an admissible measurable coloring of $\mathbb{R}^n$ so the conjecture also implies that $\chi_{P,m}(\mathbb{R}^n) = 2^n$.

Convex symmetric polytopes that tile space by translations

- Lattices give rise to such polytopes: their Dirichlet-Voronoï cells.
- In dimension 2, two combinatorial types: the rectangle and the hexagon.

- In dimension 3, there are 5 combinatorial types:

$$\mathbb{Z}^3 \qquad A_2 \perp \mathbb{Z} \qquad A_3 \qquad \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \qquad A_3^\#$$

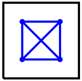
- Voronoï conjecture 1908: a translative convex polytope is the affine image of the Voronoï cell of a lattice. Proved by Delone for $n \leq 4$.

Methods

- The combinatorial bound:

$$m_P(\mathbb{R}^n) \leq \frac{\alpha(G)}{|V|}$$

- Example: the hypercube



G is the complete graph $\Rightarrow m_P(\mathbb{R}^n) = 1/2^n$

- The eigenvalue-Fourier bound: Let μ be a measure supported on ∂P,

$$m_P(\mathbb{R}^n) \leq \frac{-\min \widehat{\mu}(u)}{\widehat{\mu}(0) - \min \widehat{\mu}(u)}.$$

Recall:

$$\widehat{\mu}(u) = \int_{\partial P} e^{2i\pi(x\cdot u)} d\mu(x) \qquad \widehat{\mu}(0) = \mu(\partial P)$$

Joint work with Thomas Bellitto, Philippe Moustrou, Arnaud Pêcher (2017)

Theorem

If P tiles the plane, then

$$m_P(\mathbb{R}^2) = 1/4$$

Theorem

If P is the Dirichlet-Voronoï cell of the root lattice A_n, $n \geq 2$ then

$$m_P(\mathbb{R}^n) = 1/2^n$$

If P is the Dirichlet-Voronoï cell of the root lattice D_n, $n \geq 4$, then

$$m_P(\mathbb{R}^n) \leq 1/((3/4)2^n + n - 1)$$

$$A_n := \mathbb{Z}^{n+1} \cap \{\sum_{i=0}^{n} x_i = 0\} \qquad D_n := \{(x_1, \dots, x_n) \in \mathbb{Z}^n \; : \; \sum_{i=1}^{n} x_i = 0 \bmod 2\}$$

The Fourier-eigenvalue bound (P can be any symmetric convex body)

For all μ supported on ∂P, $\quad m_P(\mathbb{R}^n) \leq \dfrac{-\min \widehat{\mu}(u)}{\widehat{\mu}(0) - \min \widehat{\mu}(u)}$

- Let A be 1-avoiding and L-periodic. The auto-correlation function associated to A:

$$f_A(x) = \frac{1}{\mathrm{vol}(L)} \int_{\mathbb{R}^n/L} \mathbf{1}_A(x+y)\,\mathbf{1}_A(y)dy.$$

- Let $m := \min \widehat{\mu}(u)$ and

$$\nu := \mu - m\delta_{0^n}. \quad \text{We have } \widehat{\nu} = \widehat{\mu} - m \geq 0.$$

- We compute in two different ways

$$\int f_A(x)d\nu(x) = -mf_A(0^n) = -m\delta(A)$$
$$= \sum_{u \in L^\#} \widehat{f_A}(u)\widehat{\nu}(u) \geq \widehat{f_A}(0^n)\widehat{\nu}(0^n) = \delta(A)^2(\widehat{\mu}(0) - m).$$

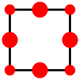
The Fourier-eigenvalue bound for polytopes
On going work with Philippe Moustrou and Sinai Robins

For all μ supported on ∂P, $\quad m_P(\mathbb{R}^n) \leq \dfrac{-\min \widehat{\mu}(u)}{\widehat{\mu}(0) - \min \widehat{\mu}(u)}$

- Without loss of generality, μ can be chosen invariant under the orthogonal group of P (by convexity argument).
- For $P = S^{n-1}$, it leaves only one possibility up to scaling: the surface measure ω.
- For other P, e.g. polytopes, lots of possibilities! (is it a good or a bad news?)
- How can we optimize over μ?
- We will see that point measures boil down to polynomial optimization problems when the points have rational coordinates (the polytope having vertices in $\mathbb{Z}^n$).
- Moreover, in this case the weights can be viewed as additional polynomial variables, so optimizing over the weights for a fixed finite support amounts again to solving a polynomial optimization problem.

A toy example

- The square



$$\mu = \tfrac{1}{4}\sum \delta_{(\pm 1,\pm 1)} + \tfrac{1}{2}\sum(\delta_{(\pm 1,0)} + \delta_{(0,\pm 1)})$$

- We have

$$\begin{aligned}
\widehat{\mu}(u) &= \frac{1}{4}\sum e^{2\pi i(\pm u_1 \pm u_2)} + \frac{1}{2}(\sum e^{2\pi i(\pm u_1)} + \sum e^{2\pi i(\pm u_2)})\\
&= \cos(2\pi u_1)\cos(2\pi u_2) + \cos(2\pi u_1) + \cos(2\pi u_2)\\
&= (\cos(2\pi u_1)+1)(\cos(2\pi u_2)+1) - 1
\end{aligned}$$

- Leading to

$$\widehat{\mu}(0) = 3, \quad \min \widehat{\mu}(u) = -1, \quad \text{bound} = \frac{1}{3+1} = \frac{1}{4}$$

The hypercube

- The centers of the k-dimensional faces are up to permutation of the coordinates: $(0, ..., 0, \pm 1, \dots, \pm 1)$ with k zeroes.
- The measure μ supported on these points weighted by $1/2^k$:

$$X_j = cos(2\pi u_j), \qquad \widehat{\mu}(u) = \prod_{j=1}^{n}(X_j + 1) - 1$$

  and has total volume $2^n - 1$ and minimum -1 leading to the sharp bound 2^{-n}.
- In general, if P is invariant under $\{\pm 1\}^n$, the Fourier transform of a measure supported on points with rational coordinates can be expressed as a polynomial in the variable $X_j = \cos(2\pi u_j/k)$ if k is a common denominator of the coordinates.

The crosspolytope $CP_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid |x_1| + \cdots + |x_n| \leq 1\}$

- We consider measures supported on:

$$\partial CP_n \cap \frac{1}{k}\mathbb{Z}^n = \{x \in \frac{1}{k}\mathbb{Z}^n \mid \ |x_1| + \cdots + |x_n| = 1\}$$

- The orbits of $kx = d$ under the action of $\{\pm 1\}^n.S_n$, are represented by the partitions of k in at most n parts:

$$d = (d_1, \dots, d_n), \quad d_1 \geq d_2 \geq \cdots \geq d_n \geq 0, \quad d_1 + \cdots + d_n = k$$

- Let us denote this set $\mathcal{P}_{k,n}$. The Fourier transform of a measure invariant under $\{\pm 1\}.S_n$ and with support contained in $\partial CP_n \cap \frac{1}{k}\mathbb{Z}^n$:

$$\widehat{\mu}(u) = \sum_{d \in \mathcal{P}_{k,n}} \lambda_d \cos(2\pi \frac{d_1 u_1}{k}) \dots \cos(2\pi \frac{d_n u_n}{k})$$

- Using the Chebyshev polynomials T_ℓ we obtain a polynomial:

$$\widehat{\mu}(u) = \sum_{d \in \mathcal{P}_{k,n}} \lambda_d T_{d_1}(X_1) \dots T_{d_n}(X_n), \qquad X_j = \cos(2\pi \frac{u_j}{k})$$

The crosspolytope

- It remains to minimize over the variables X and over the weights λ: a polynomial optimization problem that can be treated through sums of squares techniques.

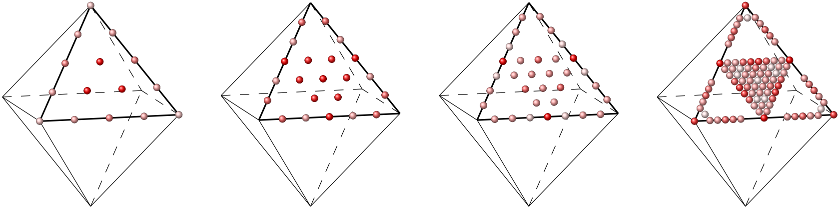
$$\min \Big\{ \sum_{d \in \mathcal{P}_{k,n}} \lambda_d T_{d_1}(X_1) \dots T_{d_n}(X_n) \ : \ -1 \leq X_j \leq 1, \ \sum \lambda_d = 2^n - 1 \Big\}$$

- Numerical results for $n = 3$:

k	Nb of pts	$\min \widehat{\mu}$	Bound
1	6	-7	0.5
2	18	-1.4	0.1666
4	42	-1.3253	0.1592
6	122	-1.3201	0.1586
8	258	-1.3195	0.1586
18	1298	-1.3156	0.1582

The optimal measures on CP_3

The distribution of weights in the numerically optimal measure for $k = 4, 6, 8, 18$:



More numerical results

- The crosspolytopes

Dimension	Division	Minimum	Bound
3	18	-1.3156	0.1582
4	4	-1.5213	0.09208
5	8	-1.9742	0.05988

- The Voronoï cells of D_n : $\|x\|_P = \max_{i \neq j} \frac{|x_i|+|x_j|}{2}$

Dimension	Division	Minimum	Bound
3	2	-1.3704	0.1638
4	2	-1.6621	0.09976
5	2	-1.86	0.0566

  The optimal support appears to be: the vertices and the middle of two vertices belonging to a common facet.

- The Voronoï cells of $D_n^{\#}$: $\|x\|_P = \max\left\{\frac{\|x\|_\infty}{2}, \frac{\|x\|_1}{n}\right\}$

Dimension	Division	Minimum	Bound
3	4	-1.4143	0.1680
5	2	-2.2202	0.06684